arXiv:1010.5304v1 [math.CT] 26 Oct 2010

Note on star-autonomous comonads

Craig Pastro

October 27, 2010

Abstract

We develop an alternative approach to star-autonomous comonads via linearly distributive categories. It is shown that in the autonomous case the notions of star-autonomous comonad and Hopf comonad coincide.

1 Introduction

Given a linearly distributive category $\mathcal{C}$, this note determines what structure is required of a comonad G on $\mathcal{C}$ so that $\mathcal{C}^G$, the category of Eilenberg-Moore coalgebras of G, is again a linearly distributive category. Furthermore, if $\mathcal{C}$ is equipped with negations (and is hence a star-autonomous category), the structure required to lift the negations to $\mathcal{C}^G$ is determined as well. This latter is equivalent to lifting star-autonomy and it is shown that the notion presented is equivalent to a star-autonomous comonad [PS09]. As a consequence of the presentation given here, it may be easily seen that any star-autonomous comonad on an autonomous category is a Hopf monad [BV07].

2 Lifting linear distributivity

Suppose $\mathcal{C}$ is a monoidal category and $G : \mathcal{C} \to \mathcal{C}$ is a comonad on $\mathcal{C}$. Recall that $\mathcal{C}^G$, the category of (Eilenberg-Moore) coalgebras of G, is monoidal if and only if G is a monoidal comonad [M02]. In this section we are interested in the structure required to lift linear distributivity to the category of coalgebras.

A linearly distributive category $\mathcal{C}$ is a category equipped with two monoidal structures $(\mathcal{C}, \star, I)$ and $(\mathcal{C}, \diamond, J)$,[1] and two compatibility natural transformations (called "linear distributions")

$$\partial_l : A \star (B \diamond C) \to (A \star B) \diamond C$$
$$\partial_r : (B \diamond C) \star A \to B \diamond (C \star A),$$

satisfying a large number of coherence diagrams [CS97].

Suppose $G = (G, \delta, \epsilon)$ is a comonad on a linearly distributive category $\mathcal{C}$ which is a monoidal comonad on $\mathcal{C}$ with respect to both $\star$ and $\diamond$, with structure

[1] For simplicity we assume that the monoidal structures are strict, although this is not necessary. Furthermore, in their original paper [CS97] the tensor products $\star$ and $\diamond$ are respectively denoted by $\otimes$ and $\diamond$, and called *tensor* and *par*, emphasizing their connection to linear logic.

maps (G, ϕ, ϕ_0) and (G, ψ, ψ_0) respectively. If, for G-coalgebras A, B, and C, the comonad G satisfies

$$
\begin{tikzcd}
GA \star (GB \diamond GC) \arrow[r, "1\star\psi"] \arrow[d, "\partial_l"'] & GA \star G(B \diamond C) \arrow[r, "\phi"] & G(A \star (B \diamond C)) \arrow[d, "\partial_l"] \\
(GA \star GB) \diamond GC \arrow[r, "\phi\diamond 1"] & G(A \star B) \diamond GC \arrow[r, "\psi"] & G((A \star B) \diamond C),
\end{tikzcd}
\tag{1}
$$

it may be seen that the morphism ∂_l becomes a G-coalgebra morphism. If G satisfies a similar axiom for ∂_r, i.e.,

$$
\begin{tikzcd}
(GB \diamond GC) \star GA \arrow[r, "\psi\star 1"] \arrow[d, "\partial_r"'] & G(B \diamond C) \star GA \arrow[r, "\phi"] & G((B \diamond C) \star A) \arrow[d, "\partial_r"] \\
GB \diamond (GC \star GA) \arrow[r, "1\diamond\phi"] & GB \diamond G(C \star A) \arrow[r, "\psi"] & G(B \diamond (C \star A)),
\end{tikzcd}
\tag{2}
$$

then ∂_r also becomes a G-coalgebra morphism. Thus,

Proposition 2.1. *Given a linearly distributive category $\mathcal{C}$ and a comonad $G : \mathcal{C} \to \mathcal{C}$ satisfying axioms* (1) *and* (2)*, the category $\mathcal{C}^G$ is a linearly distributive category.*

Example 2.2. Let $\mathcal{C}$ be a symmetric linearly distributive category and $(B, \mu, \eta, \delta, \epsilon)$ a bialgebra in $\mathcal{C}$ with respect to $\diamond$. That is, the structure morphisms are given as

$$
\begin{aligned}
\mu &: B \diamond B \to B & \qquad \delta &: B \to B \diamond B \\
\eta &: J \to B & \qquad \epsilon &: B \to J.
\end{aligned}
$$

Then, $G = B \diamond -$ is a comonad and is monoidal with respect to both $\star$ and $\diamond$. The latter by $I \cong J \diamond I \xrightarrow{\eta\diamond 1} B * I$, and the following,

$$
\begin{aligned}
(B \diamond U) \star (B \diamond V) &\xrightarrow{\quad\partial_r\quad} B \diamond (U \star (B \diamond V)) \\
&\xrightarrow{1 \diamond (1 \star c)} B \diamond (U \star (V \diamond B)) \\
&\xrightarrow{\quad 1 \diamond \partial_l\quad} B \diamond ((U \star V) \diamond B) \\
&\xrightarrow{\quad 1 \diamond c\quad} B \diamond (B \diamond (U \star V)) \\
&\xrightarrow{\quad\cong\quad} (B \diamond B) \diamond (U \star V) \\
&\xrightarrow{\quad\mu \star 1\quad} B \diamond (U \star V).
\end{aligned}
$$

Rather large diagrams, which we leave to the faith of the reader, prove that $B\diamond-$ satisfies (1) and (2), so that $\mathcal{C}^B = \mathbf{Comod}_{\mathcal{C}}(B)$, the category of comodules of B, is a linearly distributive category.

3 Lifting negations

Suppose now that $\mathcal{C}$ is a linearly distributive category equipped with negations S and S' (corresponding to ${}^{\perp}(-)$ and $(-)^{\perp}$ in [CS97]). That is, functors S, S' :

$\mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ together with the following (dinatural) evaluation and coevaluation morphisms

$$
\begin{array}{lll}
\text{(3)} & SA \star A \xrightarrow{e_A} J & A \star S'A \xrightarrow{e'_A} J \\
 & I \xrightarrow{n_A} A \diamond SA & I \xrightarrow{n'_A} S'A \diamond A,
\end{array}
$$

satisfying the four evident "triangle identities". One such is

$$\Big(A \cong I \star A \xrightarrow{n \star 1} (A \diamond SA) \star A \xrightarrow{\partial_r} A \diamond (SA \star A) \xrightarrow{1 \diamond e} A \diamond J \cong A\Big) = 1_A.$$

If $\mathcal{C}$ is equipped with such negations we say simply that $\mathcal{C}$ is a *linearly distributive category with negations.*

We are interested to lift negations to $\mathcal{C}^G$. This means we must ensure that the "negation" functors $S, S' : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ lift to functors $(\mathcal{C}^G)^{\mathrm{op}} \to \mathcal{C}^G$, and the evaluation and coevaluation morphisms are in $\mathcal{C}^G$, i.e., are G-coalgebra morphisms.

The following is essentially known from [S72].

Lemma 3.1. *A (contravariant) functor $S : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ may be lifted to a functor $\widetilde{S} : (\mathcal{C}^G)^{\mathrm{op}} \to \mathcal{C}^G$ such that the diagram*

$$
\begin{tikzcd}
(\mathcal{C}^G)^{\mathrm{op}} \arrow[r, "\widetilde{S}"] \arrow[d, "U"'] & \mathcal{C}^G \arrow[d, "U"] \\
\mathcal{C}^{\mathrm{op}} \arrow[r, "S"] & \mathcal{C},
\end{tikzcd}
$$

commutes, if and only if there is a natural transformation

$$\nu : S \to GSG$$

satisfying the following two axioms

$$
\text{(4)} \quad
\begin{tikzcd}
S \arrow[r, "\nu"] \arrow[dr, "S\epsilon"'] & GSG \arrow[d, "\epsilon_{SG}"] \\
& SG
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r, "\nu"] \arrow[dr, "\nu"'] & GSG \arrow[r, "\delta_{SG}"] & G^2SG \\
& GSG \arrow[r, "G\nu_G"] & G^2SG^2. \arrow[u, "G^2S\delta"']
\end{tikzcd}
$$

This may be viewed as a distributive law of a contravariant functor over a comonad [S72]. In this case, we say that S *may be lifted to* $\mathcal{C}^G$, and a functor $\widetilde{S} : (\mathcal{C}^G)^{\mathrm{op}} \to \mathcal{C}^G$ is defined as

$$\widetilde{S}(A, \gamma) = \big(SA,\ SA \xrightarrow{\nu} GSGA \xrightarrow{GS\gamma} GA\big) \qquad \widetilde{S}(f) = Sf.$$

(To see the reverse direction, suppose (A, γ) is a coalgebra and $\widetilde{S}$ is a functor $\mathcal{C}^G \to \mathcal{C}^G$, so that $\widetilde{S}A = (SA, \widetilde{\gamma})$ is again a coalgebra. Define

$$\nu := SA \xrightarrow{\widetilde{\gamma}} GSA \xrightarrow{GS\epsilon_A} GSGA,$$

which may be seen to satisfy the axioms in (4).) We will usually let the context differentiate between S and $\widetilde{S}$ and simply write S in both cases.

Now, suppose S and S' are equipped with natural transformations

$$\nu : S \to GSG \qquad \text{and} \qquad \nu' : S' \to GS'G.$$

such that they can be lifted to $\mathcal{C}^G$. It remains to lift the evaluation and coevaluation morphisms (3). Consider the following axioms.

(5)
$$\begin{tikzcd}
SA \star GA \arrow[r, "1\star\epsilon"] \arrow[d, "\nu\star\delta"'] & SA \star A \arrow[r, "e_A"] & J \arrow[d, "\psi_0"] \\
GSGA \star G^2A \arrow[r, "\phi"] & G(SGA \star GA) \arrow[r, "Ge_{GA}"] & GJ
\end{tikzcd}$$

(6)
$$\begin{tikzcd}
I \arrow[r, "\phi_0"] \arrow[d, "n"'] & GI \arrow[r, "Gn"] & G(A \diamond SA) \arrow[r, "G(1\diamond S\epsilon)"] & G(A \diamond SGA) \\
GA \diamond SGA \arrow[rr, "1\diamond\nu"] & & GA \diamond GSG^2A \arrow[r, "\phi"] & G(A \diamond SG^2A) \arrow[u, "G(1\diamond S\delta)"']
\end{tikzcd}$$

(7)
$$\begin{tikzcd}
GA \star S'A \arrow[r, "\epsilon\star 1"] \arrow[d, "\delta\star\nu'"'] & A \star S'A \arrow[r, "e'_A"] & J \arrow[d, "\psi_0"] \\
G^2A \star GS'GA \arrow[r, "\phi"] & G(GA \star S'GA) \arrow[r, "Ge'_{GA}"] & GJ
\end{tikzcd}$$

(8)
$$\begin{tikzcd}
I \arrow[r, "\phi_0"] \arrow[d, "n'"'] & GI \arrow[r, "Gn'"] & G(S'A \diamond A) \arrow[r, "G(S'\epsilon\diamond 1)"] & G(S'GA \diamond A) \\
S'GA \diamond GA \arrow[rr, "\nu'\diamond 1"] & & GS'G^2A \diamond GA \arrow[r, "\phi"] & G(S'G^2A \diamond A) \arrow[u, "G(S'\delta\diamond 1)"']
\end{tikzcd}$$

Proposition 3.2. *Suppose $\mathcal{C}$ is a linearly distributive category with negation, G is a monoidal comonad satisfying axioms* (1) *and* (2) *(so that $\mathcal{C}^G$ is linearly distributive), and that S and S' may be lifted to $\mathcal{C}^G$. Then, G satisfies axioms* (5), (6), (7), *and* (8) *if and only if $\mathcal{C}^G$ is a linearly distributive category with negation.*

Proof. Suppose (A,γ) is a G-coalgebra. We start by proving that axiom (5) holds if and only if $e : SA \star A \to J$ is a G-coalgebra morphism. The following diagram proves the "only if" direction,

$$\begin{tikzcd}
SA \star A \arrow[r, "\nu\star\gamma"] \arrow[d, "1\star\gamma"] \arrow[dd, bend right, "1"'] & GSGA \star GA \arrow[r, "\phi"] \arrow[d, "1\star G\gamma"'] & G(SGA \star A) \arrow[d, "G(1\star\gamma)"'] \arrow[dr, "G(S\gamma\star 1)"] & \\
SA \star GA \arrow[r, "\nu\star\delta"] \arrow[d, "1\star\epsilon"] & GSGA \star G^2A \arrow[r, "\phi"] \arrow[d, phantom, "(5)"] & G(SGA \star GA) \arrow[dr, "Ge"] & G(SA \star A) \arrow[d, "Ge"] \\
SA \star A \arrow[r, "e"] & J \arrow[rr, "\psi_0"] & & GJ,
\end{tikzcd}$$

and this next diagram the "if" direction

$$
\begin{tikzcd}
 & SA \star GA \arrow[dl, "1\star\epsilon"'] \arrow[d, "S\epsilon\star 1"] \arrow[rr, "\nu\star\delta"] & & GSGA \star G^2A \arrow[dl, "GSG\epsilon\star 1"'] \arrow[dr, "1"] \arrow[drr, bend left, "\phi"] & & \\
SA \star A \arrow[dr, "e"'] & SGA \star GA \arrow[r, "\nu\star\delta"] \arrow[d, "e"] & GSG^2A \star G^2A \arrow[rr, "GS\delta\star 1"] & & GSGA \star G^2A \arrow[r, "\phi"] & G(SGA \star GA) \arrow[d, "Ge"] \\
 & J \arrow[rrrr, "\psi_0"] & & & & GJ,
\end{tikzcd}
$$

where the bottom square commutes as e_{GA} is a G-coalgebra morphism.

Next we prove that axiom (6) holds if and only if $n : I \to A \diamond SA$ is a G-coalgebra morphism. The "only if" direction is given by

$$
\begin{tikzcd}
I \arrow[rrr, "\phi_0"] \arrow[d, "n"'] \arrow[dr, "n"] & & & GI \arrow[r, "Gn"] \arrow[d, phantom, "(6)"] & G(A \diamond SA) \arrow[d, "G(1\diamond S\epsilon)"'] \arrow[dd, bend left=60, "1"] \\
A \diamond SA \arrow[dr, "\gamma\diamond 1"'] & GA \diamond SGA \arrow[r, "1\diamond\nu"] \arrow[d, "1\diamond S\gamma"] & GA \diamond GSG^2A \arrow[r, "\phi"] \arrow[d, "1\diamond GSG\gamma"'] & G(A \diamond SG^2A) \arrow[r, "G(1\diamond S\delta)"] \arrow[d, "G(1\diamond SG\gamma)"'] & G(A \diamond SGA) \arrow[d, "G(1\diamond S\gamma)"'] \\
 & GA \diamond SA \arrow[r, "1\diamond\nu"] & GA \diamond GSGA \arrow[r, "\phi"] & G(A \diamond SGA) \arrow[r, "G(1\diamond S\gamma"] & G(A \diamond SA),
\end{tikzcd}
$$

and the "if" direction by

$$
\begin{tikzcd}
I \arrow[rrr, "\phi_0"] \arrow[d, "n"'] & & & GI \arrow[d, "Gn"'] \arrow[dr, "Gn"] & \\
GA \diamond SGA \arrow[r, "\delta\diamond\nu"] \arrow[dr, "1\diamond\nu"'] & G^2A \diamond GSG^2A \arrow[r, "\psi"] \arrow[d, "G\epsilon\diamond 1"] & G(GA \diamond SG^2A) \arrow[r, "G(1\diamond S\delta)"] \arrow[d, "G(\epsilon\diamond 1)"] & G(GA \diamond SGA) \arrow[d, "G(\epsilon\diamond 1)"'] & G(A \diamond SA) \arrow[dl, "G(1\diamond G\epsilon)"] \\
 & GA \diamond GSG^2A \arrow[r, "\psi"] & G(A \diamond SG^2A) \arrow[r, "G(1\diamond S\delta)"] & G(A \diamond SGA), &
\end{tikzcd}
$$

where the top square commutes as n_{GA} is a G-coalgebra morphism.

The remaining two axioms are proved similarly. □

4 Star-autonomous comonads

Suppose $\mathcal{C} = (\mathcal{C}, \otimes, I)$ is a star-autonomous category. A star-autonomous comonad $G : \mathcal{C} \to \mathcal{C}$ is a comonad satisfying axioms (described below) so that $\mathcal{C}^G$ becomes a star-autonomous category [PS09]. In this section we show that comonads as in Proposition 3.2 and star-autonomous comonads coincide.

We recall the definition of star-autonomous comonad [PS09], but, as it suits our needs better here, we present a more symmetric version. First recall that a star-autonomous category may be defined as a monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, I)$ equipped with an equivalence

$$S \dashv S' : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$$

such that

$$\mathcal{C}(A \otimes B, SC) \cong \mathcal{C}(A, S(B \otimes C)), \tag{9}$$

natural in $A, B, C \in \mathcal{C}$. The functor S is called the *left star operation* and S' the *right star operation*.

By the Yoneda lemma, the isomorphism in (9) determines, and is determined by, the two following "evaluation" morphisms:

$$e = e_{A,B} : S(A \otimes B) \otimes A \to SB \quad \text{and} \quad e' = e'_{B,A} : B \otimes S'(A \otimes B) \to S'A.$$

Definition 4.1. A *star-autonomous comonad* on a star-autonomous category $\mathcal{C}$ is a monoidal comonad $G : \mathcal{C} \to \mathcal{C}$ equipped with

$$\nu : S \to GSG \qquad \text{and} \qquad \nu' : S' \to GS'G,$$

satisfying (4) (i.e., S, S' may be lifted to $\mathcal{C}^G$), and this data must be such that the following four diagrams commute.

$$\begin{tikzcd} SS'G \arrow[r, "\cong"] \arrow[d, "\nu"'] & G \arrow[d, "\cong"] \\ GSGS'G \arrow[r, "GS\nu'"] & GSS' \end{tikzcd} \qquad \begin{tikzcd} S'SG \arrow[r, "\cong"] \arrow[d, "\nu'"'] & G \arrow[d, "\cong"] \\ GS'GSG \arrow[r, "GS\nu"] & GS'S \end{tikzcd}$$

$$\begin{tikzcd} S(A \otimes B) \otimes GA \arrow[r, "1\otimes\epsilon"] \arrow[d, "\nu\otimes\delta"'] & S(A \otimes B) \otimes A \arrow[r, "e_{A,B}"] & SB \arrow[d, "\nu"] \\ GSG(A \otimes B) \otimes G^2A \arrow[d, "\phi"'] & & GSGB \\ G(SG(A \otimes B) \otimes GA) \arrow[rr, "G(S\phi\otimes 1)"'] & & G(S(GA \otimes GB) \otimes GA) \arrow[u, "Ge_{GA,GB}"'] \end{tikzcd}$$

$$\begin{tikzcd} GB \otimes S'(A \otimes B) \arrow[r, "\epsilon\otimes 1"] \arrow[d, "\delta\otimes\nu'"'] & B \otimes S'(A \otimes B) \arrow[r, "e'_{B,A}"] & S'A \arrow[d, "\nu'"] \\ G^2B \otimes GS'G(A \otimes B) \arrow[d, "\phi"'] & & GS'GA \\ G(GB \otimes S'G(A \otimes B)) \arrow[rr, "G(1\otimes S'\phi)"'] & & G(GB \otimes S'(GA \otimes GB)) \arrow[u, "Ge'_{GB,GA}"'] \end{tikzcd}$$

The first two diagrams above ensure that the equivalence $S \simeq S'$ lifts to $\mathcal{C}^G$, while the latter two diagrams above respectively ensure that e and e' are G-coalgebra morphisms, so that the isomorphism (9) also lifts to $\mathcal{C}^G$.

We wish to show that star-autonomous comonads and comonads as in Proposition 3.2 coincide. It should not be surprising given the following theorem.

Theorem 4.2 ([CS97, Theorem 4.5])**.** *The notions of linearly distributive categories with negation and star-autonomous categories coincide.*

Given a star-autonomous category, identifying $\star := \otimes$ (and the units $I := I_\star = I_\otimes$) and defining

$$A \diamond B := S'(SB \star SA) \cong S(S'B \star S'A) \qquad J := SI \cong S'I \tag{10}$$

gives a linearly distributive category [CS97]. The negations of course come from S and S'. In [CS97], they consider the symmetric case, but the correspondence between linearly distributive categories with negation and star-autonomous categories holds in the noncommutative case as well.

Now, given Theorem 4.2, Proposition 3.2 says that if $\mathcal{C}$ is star-autonomous, and G is such a comonad, then $\mathcal{C}^G$ is star-autonomous. We now compare the two definitions.

Suppose now that G is a comonad on a linear distributive category $\mathcal{C}$ as in Proposition 3.2. We wish to show that it is a star-autonomous comonad. Rather than proving the axioms, it is simpler to show directly that the morphisms under consideration are G-coalgebra morphisms. To this end, the equivalence $S \simeq S'$ is given by the equations

$$A \cong I \star A \xrightarrow{n'_{SA} \star 1} (S'SA \diamond SA) \star A \xrightarrow{\partial_r} S'SA \diamond (SA \star A) \xrightarrow{1 \diamond n} S'SA \diamond J \cong S'SA$$

and

$$S'SA \cong I \star S'SA \xrightarrow{n_A \star 1} (A \diamond SA) \star S'SA \xrightarrow{\partial_r} A \diamond (SA \star S'SA) \xrightarrow{1 \diamond e'_{SA}} A \diamond J \cong A,$$

and $e_{A,B}$ and $e'_{B,A}$ are respectively defined as

$$\begin{tikzcd}
S(A \star B) \star A \arrow[r, dotted, "e_{A,B}"] \arrow[d, "\cong"'] & SB \\
S(A \star B) \star A \star I \arrow[d, "1 \star 1 \star n"'] & J \diamond SB \arrow[u, "\cong"'] \\
S(A \star B) \star A \star (B \diamond SB) \arrow[r, "\partial_l"] & (S(A \star B) \star A \star B) \diamond SB \arrow[u, "e_{A \star B} \diamond 1"']
\end{tikzcd}$$

$$\begin{tikzcd}
B \star S'(A \star B) \arrow[r, dotted, "e'_{B,A}"] \arrow[d, "\cong"'] & SB \\
I \star B \star S'(A \star B) \arrow[d, "n' \star 1 \star 1"'] & S'A \diamond J \arrow[u, "\cong"'] \\
(S'A \diamond A) \star B \star S'(A \star B) \arrow[r, "\partial_r"] & S'A \diamond (A \star B \star S'(A \star B)) \arrow[u, "1 \diamond e'_{A \star B}"']
\end{tikzcd}$$

In the situation of Proposition 3.2, we see that all four of these morphisms are given as composites of G-coalgebra morphisms, and thus, are G-coalgebra morphisms themselves. Therefore, G is a star-autonomous comonad.

In the other direction suppose G is a star-autonomous comonad on a star-autonomous category $\mathcal{C}$. It is similar to show that it is a comonad satisfying the requirements of Proposition 3.2. Using the identifications in (10), the two linear distributions are defined as follows.

$$\begin{tikzcd}
A \star (B \diamond C) \arrow[rr, dotted, "\partial_l"] \arrow[d, "\cong"'] & & (A \star B) \diamond C \\
A \otimes S'(SC \otimes SB) \arrow[dr, "1 \otimes S'(1 \otimes e)"'] & & S'(SC \otimes S(A \otimes B)) \arrow[u, "\cong"'] \\
& A \otimes S'(SC \otimes S(A \otimes B) \otimes A) \arrow[ur, "e'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(B \diamond C) \star A \arrow[rr, dotted, "\partial_r"] \arrow[d, "\cong"'] & & B \diamond (C \star A) \\
S(S'C \otimes S'B) \otimes A \arrow[dr, "S(e' \otimes 1) \otimes 1"'] & & S(S'(C \otimes A) \otimes S'B) \arrow[u, "\cong"'] \\
& S(A \otimes S'(C \otimes A) \otimes S'B) \otimes A \arrow[ur, "e"'] &
\end{tikzcd}$$

The evaluation maps e_A and e'_A are defined as $e_{A,I}$ and $e'_{A,I}$, and the coevaluation maps n_A and n'_A as

$$n_A = \left(I \cong SS'I \xrightarrow{Se'_{A,I}} S(A \otimes S'A) = A \diamond SA \right)$$

$$n'_A = \left(I \cong S'SI \xrightarrow{S'e_{A,I}} S'(SA \otimes A) = S'A \diamond A \right)$$

Again, each morphism is a G-coalgebra morphism, or composite thereof, and therefore is itself a G-coalgebra morphism.

Thus, both notions coincide, and we will simply call either notion a *star-autonomous comonad*, and let context differentiate the axiomatization.

Example 4.3. Any Hopf algebra H in a star-autonomous category $\mathcal{C}$ gives rise to a star-autonomous comonad $H \otimes - : \mathcal{C} \to \mathcal{C}$. See [PS09, pg. 3515] for details.

Example 4.4. If $\mathcal{C}$ is a symmetric closed monoidal category with finite products, then we may apply the Chu construction [B79] to produce a star-autonomous category $\mathrm{Chu}(\mathcal{C})$. $\mathcal{C}$ fully faithfully embeds into $\mathrm{Chu}(\mathcal{C})$,

$$\mathcal{C} \hookrightarrow \mathrm{Chu}(\mathcal{C})$$

and this functor is strong symmetric monoidal. Thus, any Hopf algebra in $\mathcal{C}$ becomes a Hopf algebra in $\mathrm{Chu}(\mathcal{C})$, and thus, an example of a star-autonomous comonad.

5 The compact case $\star = \diamond$

If $\mathcal{C}$ is a linearly distributive category with negation for which $\star = \diamond$ (and thus, $I = J$), then $\mathcal{C}$ is an autonomous (= rigid) category. The functor S provides left duals, while S' provides right duals. It is not hard to see that in this case, any star-autonomous monad G (after dualizing) is a Hopf monad [BV07]. Set $\star = \diamond$ and $I = J$ and dualize axioms (5), (6), (7), and (8). They correspond in [BV07] to axioms (23), (22), (21), and (20) respectively. (In their notation ${}^\vee(-) = S$ and $(-)^\vee = S'$.) Therefore, we have:

Proposition 5.1. *Star-autonomous monads on autonomous categories are Hopf monads.*

Acknowledgements

I would like to thank Robin Cockett and Masahito Hasegawa for their valuable suggestions.

References

[B79] Michael Barr. $*$-Autonomous categories, Volume 752 of Lecture Notes in Mathematics. Springer, Berlin, 1979. With an appendix by Po Hsiang Chu.

[BV07] Alain Bruguières and Alexis Virelizier. Hopf monads, Advances in Mathematics 215 no. 2 (2007) 679–733.

[CS97] J.R.B. Cockett and R.A.G. Seely. Weakly distributive categories, Journal of Pure and Applied Algebra 114 (1997) 133–173. Corrected version available from the second authors webpage.

[M02] I. Moerdijk. Monads on tensor categories, Journal of Pure and Applied Algebra 168 (2002) 189–208.

[PS09] Craig Pastro and Ross Street. Closed categories, star-autonomy, and monoidal comonads, Journal of Algebra 321 no. 11 (2009) 3494–3520.

[S72] Ross Street. The formal theory of monads, Journal of Pure and Applied Algebra 2 no. 2 (1972) 149–168.

Department of Mathematics, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan
craig@math.kyushu-u.ac.jp